2.3) Energy loss in a damped harmonic oscillator (DHO)

- In the DHO, mechanical energy is dissipated as heat. Energy is not conserved (if ignore heat).

- How does energy change in time?

$$E(t) = K(t) + U(t) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

- For light damping ($\gamma^2/4 \ll \omega_0^2$) $\omega \simeq \omega_0$

$$\rightarrow x = A_0 e^{-\gamma t/2} \cos\omega_0 t$$

$$v = \frac{dx}{dt} = -A_0 \omega_0 e^{-\gamma t/2}\left[\sin\omega_0 t + \underbrace{(\gamma/2\omega_0)\cos\omega_0 t}\right]$$

- Since light damping, $\gamma/2 \ll \omega_0$, so is small

$$\rightarrow v \simeq -A_0 \omega_0 e^{-\gamma t/2} \sin\omega_0 t$$

Then: $E = \frac{1}{2}A_0^2 e^{-\gamma t}(m\omega_0^2 \sin^2\omega_0 t + k\cos^2\omega_0 t)$

$$\omega_0^2 = \frac{k}{m} \longrightarrow E = \frac{1}{2}kA_0^2 e^{-\gamma t}$$

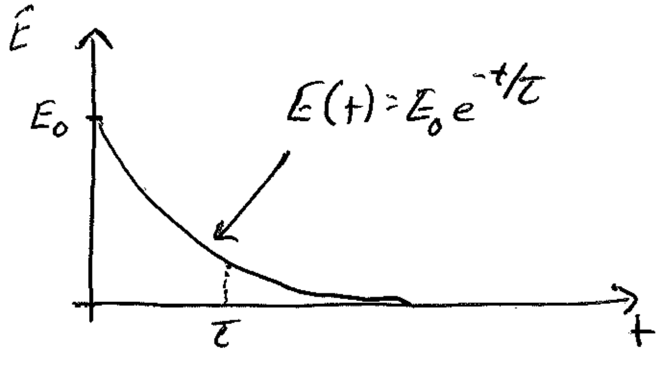
γ: decay rate

$$\Rightarrow \boxed{E(t) = E_0 e^{-\gamma t}}$$

total initial energy at $t=0$

• Can also define $\boxed{\tau = \frac{1}{\gamma}}$ → $E(t) = E_0 e^{-t/\tau}$

→ "decay time" or "time constant" or "lifetime"



• Energy of oscillator dissipates b/c it does work against the damping force at rate $\frac{dE}{dt}$ = (damping force) × (velocity)

• We can see this by $\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right)$

$$\to = mv\frac{dv}{dt} + kx\frac{dx}{dt} = \underbrace{(ma + kx)}_{\neq 0 \text{ b/c of damping } (=0 \text{ for SHO})} v = \underbrace{(-bv)}_{F_d} v$$

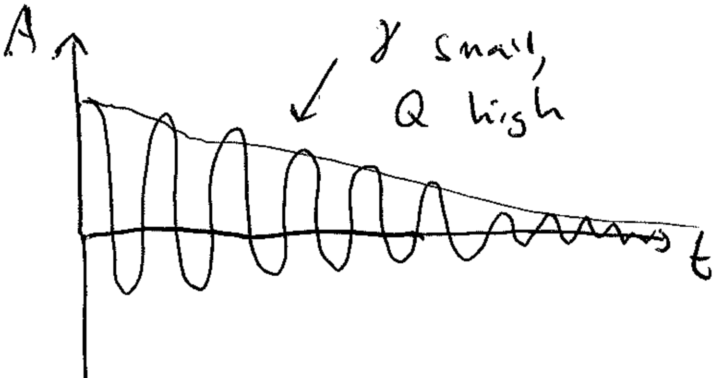
//

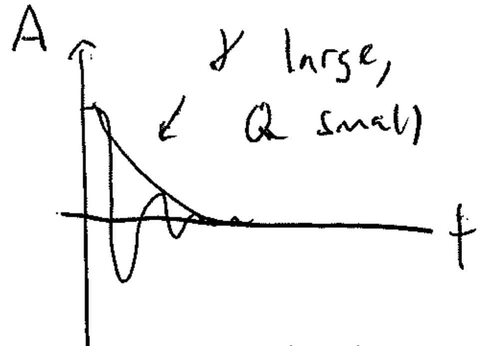
Quality factor (Q)

↳ Figure of merit to quantify "how good" an oscillator is: $\boxed{Q = \frac{\omega_0}{\gamma}}$

limit $\gamma \to 0$

$Q \to \infty$





• Q is dimensionless and can be applied to any oscillator (classical or quantum)

· Alternative interpretation using energy:

Consider the energy of a lightly damped oscillator one period apart:

$$E_1 = E_0 e^{-\gamma t} \quad ; \quad E_2 = E_0 e^{-\gamma(t+T)}$$

$$\rightarrow \frac{E_2}{E_1} = e^{-\gamma T}$$ · For light damping, $\gamma T << 1$

· Expand this: $e^{x} \approx \underbrace{1 + x}_{\text{keep these}} + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots$

$$\rightarrow \frac{E_2}{E_1} = e^{-\gamma T} \simeq 1 - \gamma T$$

· Take $\frac{E_1 - E_2}{E_1} \simeq \gamma T \simeq \frac{2\pi\gamma}{\omega_0} = \frac{2\pi}{Q} =$ Fractional energy change per cycle

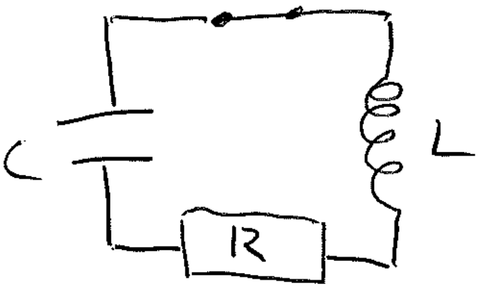
$$\rightarrow Q = \frac{\text{energy stored in oscillator}}{\text{energy dissipated per cycle per radian}} \quad \left[\frac{E_1}{(E_1 - E_2)/2\pi} \right]$$

· Can rewrite eq. of motion as: $\frac{d^2x}{dt^2} + \frac{\omega_0}{Q}\frac{dx}{dt} + \omega_0^2 x = 0$

& $\omega = \omega_0 \left(1 - \frac{1}{4Q^2}\right)^{1/2}$. For $Q \sim 5$, $\frac{\omega_0 - \omega}{\omega} \simeq 0.5\%$

· Typical Q values: 1) Clock pendulum $Q \sim 75$ 2) violin string $Q \sim 10^3$ 3) Quartz crystal: $Q \sim 10^6$

2.4 Damped electrical oscillations



- In circuits, resistance impedes flow of current. From Kirchhoff's law:

$$V_C + V_R + V_L = 0$$

$$\rightarrow L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

- Similar form to mechanical oscillator:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

- Using correspondence: $q \leftrightarrow x$, $L \leftrightarrow m$, $\frac{1}{C} \leftrightarrow k$

$$R/L \leftrightarrow \gamma$$

- Can transfer solution from mechanical case:

$$q(t) = q_0 e^{-Rt/2L} \cos\left[\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)^{1/2} t\right] \quad \left(\frac{R^2}{4L^2} < \frac{1}{LC} \text{ for light damping}\right)$$

- Since $V_C = \frac{q}{C}$

$$\rightarrow V_C = V_0 e^{-Rt/2L} \cos\left[\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)^{1/2} t\right]$$

- $$\omega^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$$

$$\frac{R^2}{4L^2} \gg \frac{1}{LC}$$ Heavy damping

$$\frac{R^2}{4L^2} = \frac{1}{LC}$$ critical damping

$$\frac{R^2}{4L^2} \ll \frac{1}{LC}$$ light damping

- We find that Quality factor $\boxed{Q = \frac{1}{R}\sqrt{\frac{L}{C}}}$

→ Resistance R clearly negatively impacts Q, but so does ration $\frac{L}{C}$

- ex: $L = 10\,mH$, $C = 2.5\,nF$, $R = 10\,\Omega$ ⇒ $Q = 200$